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Computation of periodic Green's functions of Stokes flow

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Abstract. Methods of computing periodic Green's functions of Stokes flow representing the flow due to triply-, doubly-, and singly-periodic arrays of three-dimensional or two-dimensional point forces are reviewed, developed, and discussed with emphasis on efficient numerical computation. The standard representation in terms of Fourier series requires a prohibitive computational effort for use with singularity and boundary-integral-equation methods; alternative representations based on variations of Ewald's summation method involving various types of splitting between physical and Fourier space with partial sums that decay in a Gaussian or exponential manner, allow for efficient numerical computation. The physical changes undergone by the flow in deriving singly- and doubly-periodic Green's functions from their triply-periodic counterparts are considered.

1. Introduction

The flow due to a periodic array of point forces and associated Green's functions of Stokes flow are fundamental building blocks that allow us to analyze and synthesize a variety of spatially-periodic flows using singularity and boundary-integral-equation methods [1]. The origin of these methods may be traced back to the reciprocal theorem and the boundary-integral representation presented in a seminal paper by Lorentz [2]. Examples of applications include the flow of periodic suspensions of solid particles and liquid drops in infinite space and inside tubes [3–5], the flow past periodic arrangements of obstacles representing ordered or random porous media, and the flow past model filters, membranes, and screens [6,7].

Despite its conceptual simplicity, the flow due to a periodic array of point forces has resisted efficient computation for quite some time. The central mathematical difficulty lies in the fact that the flow due to a single point force in three dimensions decays like $1/r$, and this makes the direct summation over a periodic array divergent. Similar but more severe difficulties are encountered in two dimensions. A resolution emerges by abandoning the method of direct summation and solving the periodically-forced equations of Stokes flow demanding appropriate periodic conditions for the velocity and pressure. The first computation of this type was done by Hasimoto [8], who derived the flow due to a triply-periodic array of three-dimensional point forces and the flow due to a doubly-periodic array of two-dimensional point forces in the form of Fourier series. His solution and its extensions to doubly-periodic and singly-periodic arrays will be discussed in Section 2 of the present paper.

Unfortunately, the Fourier-series representation converges slowly, requiring a prohibitive amount of effort for use in large scale numerical simulation [3–5]. Realizing this difficulty, Hasimoto [8] devised a fast summation method based on Ewald's original formulation. The idea is to replace the Fourier series with two complementary, rapidly converging sums, one over the physical lattice and the second over the wave-number reciprocal lattice. Ishii [9], and

Sangani and Behl [7], extended the Hasimoto–Ewald summation method to planar arrays of three-dimensional point forces.

A different summation method that circumvents a great deal of analytical work was proposed by Beenakker [10] for triply-periodic arrays. His method is based on a direct decomposition of the stokeslet, before it is summed over the periodic lattice. The final result is a sum in real space and a sum in reciprocal space which are similar but not identical to those derived by Hasimoto [8]. In Section 3 we discuss Beenakker's method and its extensions to doubly-periodic arrays of three-dimensional point forces. The latter produces complicated algebraic expressions involving the error function, whose computation incurs a substantial cost. To circumvent this difficulty, in Section 4 we develop an alternative fast summation method that is based on a different decomposition of the stokeslet. The method yields partials sums that converge at a rate that is slower than that of Beenakker's decomposition; its advantage is that it involves relatively simple algebraic expressions that expedite the numerical computation.

In Section 5 we discuss the analogous computation of the flow due to arrays of two-dimensional point forces. Previous authors have developed methods based on Ewald's original formulation which result in expressions involving the exponential integral [6,8,11]. We develop an alternative representation that arises by integrating the Green's function corresponding to a three-dimensional array of three-dimensional point forces in the direction of one base vector to obtain a continuous distribution in the direction of integration, and we discuss its numerical implementation.

In recent years, a considerably body of work has been devoted towards developing efficient summation methods pertinent to general types of scalar potentials encountered in molecular dynamics (see, for example [12]). The present paper contributes a parallel development in the area of low-Reynolds-number fluid mechanics.

2. Arrays of three-dimensional point forces: Fourier series solution

We begin by deriving the Fourier-series representation of the Green's function corresponding to a periodic array of three-dimensional point forces. Our main goal in this section is to establish a point of reference for the development of fast summation methods which will be considered in the subsequent two sections.

2.1. Triply-periodic arrays

Consider the flow due to a three-dimensional periodic array of point forces with identical strengths $\mathbf{g}$ placed at the vertices of a three-dimensional lattice in infinite space, where one of the point forces is located at the point $\mathbf{x}_0$. The instantaneous structure of the lattice can be described in terms of the three base vectors, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, which are oriented according to the right-handed rule, so that the nth point force is located at the point $\mathbf{x}_0 + \mathbf{X}_n$, where $\mathbf{X}_n = i_1\mathbf{a}_1 + i_2\mathbf{a}_2 + i_3\mathbf{a}_3$ and i_1, i_2, i_3 are three integers, as depicted in Fig. 1.

The associated velocity $\mathbf{u}$ and pressure p satisfy the continuity equation $\nabla \cdot \mathbf{u} = 0$ and the periodically-forced three-dimensional Stokes equation

$$-\nabla p(x) + \mu\nabla^2\mathbf{u}(\mathbf{x}) + \mathbf{g}\sum_n \delta_3(\hat{\mathbf{x}}_n) = 0, \tag{2.1}$$

where μ is the viscosity of the fluid, $\hat{\mathbf{x}}_n = \mathbf{x} - \mathbf{x}_0 - \mathbf{X}_n$, and δ_3 is the three-dimensional delta function.

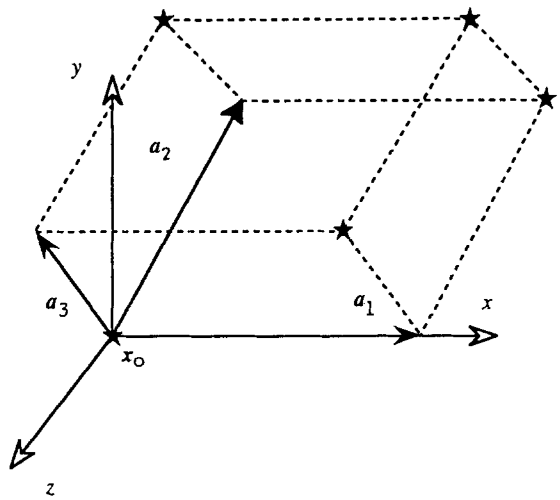

Fig. 1. A three-dimensional periodic array of point forces with identical strengths placed at the vertices of a three-dimensional lattice in infinite space. One of the point forces is located at the point $\mathbf{x}_0$.

Following standard practice, we introduce the three-dimensional periodic Green's function $\mathbf{G}^{3D3P}$ and corresponding pressure vector $\mathbf{p}^{3D3P}$ defined in terms of the equations $u_j = 1/(8\pi\mu)G_{jm}^{3D3P}g_m$ and $p = 1/(8\pi)p_m^{3D3P}g_m$. Substituting these definitions into (2.1) and the continuity equation and discarding the arbitrary constant $\mathbf{g}$ we obtain

$$-\frac{\partial p_m^{3D3P}}{\partial x_j} + \nabla^2 G_{jm}^{3D3P} + 8\pi\delta_{jm}\sum_n \delta_3(\hat{\mathbf{x}}_n) = 0, \qquad \frac{\partial G_{jm}^{3D3P}}{\partial x_j} = 0. \tag{2.2}$$

Next, following Hasimoto [8], we exploit the periodicity of the flow and expand $\mathbf{G}^{3D3P}$ and pressure gradient $\nabla\mathbf{p}^{3D3P}$ in complex Fourier series, writing

$$G_{jm}^{3D3P} = \sum_\lambda \hat{G}_{jm\lambda}^{3D3P}\exp(-i\mathbf{k}_\lambda\cdot\hat{\mathbf{x}}_0), \qquad \frac{\partial p_m^{3D3P}}{\partial x_j} = \sum_\lambda \hat{p}_{jm\lambda}^{3D3P}\exp(-i\mathbf{k}_\lambda\cdot\hat{\mathbf{x}}_0), \tag{2.3}$$

where $\mathbf{k}_\lambda = j_1\mathbf{b}_1 + j_2\mathbf{b}_2 + j_3\mathbf{b}_3$ are the reciprocal lattice points residing in the wave number space, j_1, j_2, j_3 are three integers, and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are the reciprocal base vectors defined as

$$\mathbf{b}_1 = \frac{2\pi}{\tau}\mathbf{a}_2\times\mathbf{a}_3, \quad \mathbf{b}_2 = \frac{2\pi}{\tau}\mathbf{a}_3\times\mathbf{a}_1, \quad \mathbf{b}_3 = \frac{2\pi}{\tau}\mathbf{a}_1\times\mathbf{a}_2 \tag{2.4}$$

$\tau = (\mathbf{a}_1\times\mathbf{a}_2)\cdot\mathbf{a}_3$ is the volume of a periodic cell in physical space. Note that the physical and reciprocal lattice vectors satisfy the equation $\mathbf{X}_n\cdot\mathbf{k}_\lambda = 2\pi m$ where m is an integer. Furthermore, we use Parseval's identity to write

$$\sum_n \delta_3(\hat{\mathbf{x}}_n) = \frac{1}{\tau}\sum_\lambda \exp(-i\mathbf{k}_\lambda\cdot\hat{\mathbf{x}}_0). \tag{2.5}$$

Substitution of (2.3) and (2.5) in (2.2) and grouping similar Fourier coefficients yields the algebraic system of equations

$$-\hat{p}_{jm\lambda}^{3D3P} - |\mathbf{k}_\lambda|^2\hat{G}_{jm\lambda}^{3D3P} + \frac{8\pi}{\tau}\delta_{jm} = 0, \qquad k_{\lambda j}\hat{G}_{jm\lambda}^{3D3P} = 0 \tag{2.6}$$

the solution of which is

$$\hat{G}^{3D3P}_{jm0} = 0, \qquad \hat{p}^{3D3P}_{jm0} = \frac{8\pi}{\tau}\delta_{jm} \tag{2.7a}$$

for $|\mathbf{k}_0| = 0$, and

$$\hat{G}^{3D3P}_{jm\lambda} = \frac{8\pi}{\tau}\frac{1}{|\mathbf{k}_\lambda|^2}\left(\delta_{jm} - \frac{k_{j\lambda}k_{m\lambda}}{|\mathbf{k}_\lambda|^2}\right), \qquad \hat{p}_{jm\lambda} = \frac{8\pi}{\tau}\frac{k_{j\lambda}k_{m\lambda}}{|\mathbf{k}_\lambda|^2} \tag{2.7b}$$

for $|\mathbf{k}_\lambda| \neq 0$. The zeroth-order coefficients shown in (2.7a) are responsible for the onset of a pressure gradient in the direction of the point forces. Substituting (2.7a,b) in the first equation of (2.3) yields the Green's function in the form of a Fourier series [8]

$$\begin{aligned} G^{3D3P}_{jm} &= \frac{8\pi}{\tau}\sum_{\substack{\lambda \\ |\mathbf{k}_\lambda|\neq 0}} \frac{1}{|\mathbf{k}_\lambda|^2}\left(\delta_{jm} - \frac{k_{j\lambda}k_{m\lambda}}{|\mathbf{k}_\lambda|^2}\right)\exp(-i\mathbf{k}_\lambda\cdot\hat{\mathbf{x}}_0) \\ &= \frac{8\pi}{\tau}\sum_{\substack{\lambda \\ |\mathbf{k}_\lambda|\neq 0}} \left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m}\right) S^{3D3P}(\mathbf{k}_\lambda, \hat{\mathbf{x}}_0). \end{aligned} \tag{2.8}$$

For brevity of notation we have introduce the generating function

$$S^{3D3P}(\mathbf{k},\mathbf{x}) = \frac{\exp(-i\mathbf{k}\cdot\mathbf{x})}{|\mathbf{k}|^4}. \tag{2.9}$$

The associated pressure gradient is given by

$$\begin{aligned} \frac{\partial p^{3D3P}_m}{\partial x_j} &= \frac{8\pi}{\tau}\delta_{jm} + \frac{8\pi}{\tau}\sum_{\substack{\lambda \\ |\mathbf{k}_\lambda|\neq 0}} \frac{k_{j\lambda}k_{m\lambda}}{|\mathbf{k}_\lambda|^2}\exp(-i\mathbf{k}_\lambda\cdot\hat{\mathbf{x}}_0) \\ &= \frac{8\pi}{\tau}\delta_{jm} - \frac{8\pi}{\tau}\sum_{\substack{\lambda \\ |\mathbf{k}_\lambda|\neq 0}} \frac{\partial}{\partial x_j \partial x_m}\nabla^2 S^{3D3P}(\mathbf{k}_\lambda, \hat{\mathbf{x}}_0). \end{aligned} \tag{2.10}$$

One interesting consequence of the spatial periodicity of the generating function S^{3D3P} is that the flow rate across a plane that is perpendicular to one of the base lattice vectors in physical space vanishes. This becomes evident by observing that the component of the velocity normal to such a plane is proportional to the two-dimensional Laplacian of S^{3D3P}, written with respect to the two coordinates that vary in that plane, using the divergence theorem to convert the integral of the normal component of the velocity over a periodic area to a line integral of the normal derivative of S^{3D3P} around its boundary, and then noting that the line integral vanishes because of the periodicity of S^{3D3P}.

2.2. Doubly-Periodic Arrays

To obtain a doubly-periodic array of three-dimensional point forces located in the plane of the base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, we set $\mathbf{a}_3$ perpendicular to the plane of $\mathbf{a}_1$ and $\mathbf{a}_2$, write $\tau = AL$ where $A = |\mathbf{a}_1 \times \mathbf{a}_2|$ is the area of one planar cell and $L = |\mathbf{a}_3|$, and find

$$\mathbf{b}_1 = \frac{2\pi}{A}\mathbf{a}_2 \times \mathbf{e}_3, \quad \mathbf{b}_2 = \frac{2\pi}{A}\mathbf{e}_3 \times \mathbf{a}_1, \quad \mathbf{b}_3 = \frac{2\pi}{L}\mathbf{e}_3, \tag{2.11}$$

where $\mathbf{e}_3$ is the unit vector in the direction of $\mathbf{a}_3$. In the limit, as L tends to infinity, (2.8) yields the new Green's function

$$G_{jm}^{3D2P} = \frac{4\pi}{A} \sum_{\substack{\lambda \\ |\mathbf{l}_\lambda| \neq 0}} \left(-\delta_{jm} \nabla^2 + \frac{\partial}{\partial x_j \partial x_m} \right) S^{3D2P}(\mathbf{l}_\lambda, \hat{\mathbf{x}}_0), \tag{2.12}$$

where $\mathbf{l}_\lambda = j_1 \mathbf{b}_1 + j_2 \mathbf{b}_2$ are the reciprocal lattice points lying in the plane of $\mathbf{a}_1$ and $\mathbf{a}_2$, and j_1, j_2 are two integers. The new generating function S^{3D2P} is given by

$$S^{3D2P}(\mathbf{l}, \mathbf{x}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\exp[-i(\mathbf{l} + \eta \mathbf{e}_3) \cdot \mathbf{x}]}{|\mathbf{l} + \eta \mathbf{e}_3|^4} \, \mathrm{d}\eta = \tfrac{1}{2} \frac{1 + \rho}{|\mathbf{l}|^3} \exp[-i\mathbf{l} \cdot \mathbf{x} - \rho], \tag{2.13}$$

where $\rho = |\mathbf{l}| \, |\mathbf{e}_3 \cdot \mathbf{x}|$ [6,9]. The integrals are evaluated with the help of ref. [13, p. 410].

The corresponding pressure gradient is found by replacing the first term on the right-hand side of (2.10) with a new term involving the one-dimensional delta function, δ_1, operating in the plane of the array as shown below, and integrating the last term as in (2.13) [10, p. 406], yielding

$$\frac{\partial p_m^{3D2P}}{\partial x_j} = \frac{8\pi}{A} \delta_{jm} \delta_1(\hat{x}_{0_3}) - \frac{4\pi}{A} \sum_{\substack{\lambda \\ |\mathbf{l}_\lambda| \neq 0}} \frac{\partial}{\partial x_j \partial x_m} \nabla^2 S^{3D2P}(\mathbf{l}_\lambda, \hat{\mathbf{x}}_0). \tag{2.14}$$

It is important to note that, in deriving the doubly-periodic array from the triply-periodic array, we have abandoned the summation over the one-dimensional array corresponding to the null wave number $|\mathbf{l}_\lambda| = 0$ as shown in (2.12) and (2.14); if we did not, the associated integral in (2.13) would have exhibited a singular behaviour. As a result of this modification, both the velocity and pressure gradient decay at an exponential rate far from the array.

There is a penalty that we pay by requiring that the velocity and pressure vanish far from the array: when the point forces are oriented normal to the array, the pressure undergoes a discontinuity of magnitude $[1/(\mu A)]\mathbf{g}$ across the plane of the array; when the point forces are oriented parallel to the array, the shear stress undergoes a corresponding discontinuity of same magnitude. This behaviour becomes evident by noting that the Laplacian of the generating function given in (2.13) represents the steady temperature field due to the combined action of a doubly-periodic array of point sources of heat, and a continuous distribution of point sinks of heat distributed in the plane of the array, so that the net rate of transport of heat across a period of the array is equal to zero [9].

To demonstrate the discontinuity in shear stress when the point forces are parallel to the array, we differentiate (2.12) and use the identity

$$\sum_n \delta_2(\hat{\mathbf{x}}_n) = \frac{1}{A} \sum_\lambda \exp(-i\mathbf{l}_\lambda \cdot \hat{\mathbf{x}}_0), \tag{2.15}$$

where δ_2 is the two-dimensional delta function operating in the plane of the array, to obtain

$$\pm \left(\frac{\partial G_{11}}{\partial x_3} \right)_{\hat{x}_{0_3} = 0^\pm} = \pm \left(\frac{\partial G_{22}}{\partial x_3} \right)_{\hat{x}_{0_3} = 0^\pm} = \frac{4\pi}{A} - 4\pi \sum_n \delta_2(\hat{\mathbf{x}}_n). \tag{2.16}$$

The second term on the right-hand side of (2.16) represents the effect of the point forces, whereas the first term demonstrates the discontinuous behaviour.

In order to obtain a Green's function with a continuous stress field we work as follows. When the point forces are directed normal to their plane, we enhance the pressure field with a step function that suffers an appropriate jump. When the point forces are directed parallel to the array, we introduce a simple shear flow, with an objective to annihilate the discontinuity in the slope of the velocity across the plane of the point forces. This is effected by enhancing the diagonal components of the Green's function with a symmetric shear flow, thus obtaining the *regularized* form

$$G_{jm}^{3D2P-R} = G_{jm}^{3D2P} - \frac{4\pi}{A}\delta_{jm}|\hat{x}_{0_3}|, \tag{2.17}$$

where $j, m = 1, 2$. The velocity field associated with (2.17) and its derivatives are continuous and differentiable throughout the whole space, except at the location of the point forces.

2.3. Singly-periodic array

To obtain a singly-periodic array of three-dimensional point forces arranged along the base vector $\mathbf{a}_1$, we begin from the triply-periodic array depicted in Fig. 1, set $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ perpendicular to one another, and write $\tau = aL^2$ where $a = |\mathbf{a}_1|$ is the separation of the point forces and $L = |\mathbf{a}_2| = |\mathbf{a}_3|$. We thus find

$$\mathbf{b}_1 = \frac{2\pi}{a}\mathbf{e}_1, \quad \mathbf{b}_2 = \frac{2\pi}{L}\mathbf{e}_2, \quad \mathbf{b}_3 = \frac{2\pi}{L}\mathbf{e}_3, \tag{2.18}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors in the directions of $\mathbf{a}_1$ and $\mathbf{a}_2$. Letting L tend to infinity reduces (2.8) to

$$G_{jm}^{3D1P} = \frac{4\pi}{a}\sum_{\substack{\lambda \\ |\mathbf{m}_\lambda|\neq 0}} \left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m}\right) S^{3D1P}(\mathbf{m}_\lambda, \hat{\mathbf{x}}_0), \tag{2.19}$$

where $\mathbf{m}_\lambda = \lambda\mathbf{b}_1$ are the reciprocal lattice points and λ is an integer. The corresponding generating function is given by

$$S^{3D1P}(\mathbf{m}, \mathbf{x}) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp[-i(\mathbf{m} + \xi\mathbf{e}_2 + \eta\mathbf{e}_3)\cdot\mathbf{x}]}{|\mathbf{m} + \xi\mathbf{e}_2 + \eta\mathbf{e}_3|^4}\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{2.20}$$

Expressing the variables of integration in plane polar coordinates (r, θ) and performing the integration with respect to the polar angle θ yields

$$\begin{aligned} S^{3D1P}(\mathbf{m}, \mathbf{x}) &= \frac{2}{\pi}\exp(-i\mathbf{m}\cdot\mathbf{x})\int_0^{\infty} \frac{J_0(\sigma r)}{(|\mathbf{m}|^2 + r^2)^2} r\, \mathrm{d}r \\ &= \frac{\sigma}{\pi}\frac{K_1(\sigma|\mathbf{m}|)}{|\mathbf{m}|}\exp(-i\mathbf{m}\cdot\mathbf{x}), \end{aligned} \tag{2.21}$$

where σ is the distance of the point $\mathbf{x}$ from the line of the point forces [13, pp. 482, 686].

Using the asymptotic expansion of the modified Bessel function K_1 for large values of its argument $\sigma|\mathbf{m}|$ [14, p. 378] we obtain the far-field behaviour

$$S^{3D1P} \approx \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{|\mathbf{m}|^{3/2}}\exp(-i\mathbf{m}\cdot\hat{\mathbf{x}}_0 - \sigma|\mathbf{m}|), \tag{2.22}$$

which demonstrates that the flow far from the array decays at an exponential rate.

To assess the behaviour of the flow in the vicinity of the array, which is located at $\sigma = 0$, we use the asymptotic expansion $K_1(x) = 1/x + (x/2)\ln(x/2) + \cdots$ [14, p. 375] and obtain

$$S^{3D1P}(\mathbf{m}, \mathbf{x}) \approx \frac{1}{\pi}\left(\frac{1}{|\mathbf{m}|^2} + \frac{\sigma^2}{2}\ln\left(\tfrac{1}{2}\sigma|\mathbf{m}|\right)\right)\exp(-i\mathbf{m}\cdot\mathbf{x}), \tag{2.23}$$

which reveals a logarithmic singularity at the axis. Substituting (2.23) in (2.19), switching the differential operators with the sum, and using the identity

$$\sum_n \delta_1(\hat{\mathbf{x}}_n) = \frac{1}{a}\sum_\lambda \exp(-i\mathbf{m}_\lambda\cdot\hat{\mathbf{x}}_0) \tag{2.24}$$

gives the asymptotic behaviour

$$G^{3D1P}_{jm} \approx 2\left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m}\right)(-\sigma^2\ln\sigma). \tag{2.25}$$

The generating function $\sigma^2(\ln\sigma - 1)$ produces (a) Stokes flow due to a two-dimensional point force pointing normal to the array ([1], p. 60), and (b) unidirectional flow with vanishing pressure gradient when the point forces are parallel to the array. To obtain a regularized Green's function with a non-singular behaviour at the axis, we set

$$G^{3D1P-R}_{jm} = G^{3D1P}_{jm} + 2\left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m}\right)[\sigma^2(\ln\sigma - 1)]. \tag{2.26}$$

The singular behaviour of the last term annihilates that of the first term on the right-hand side yielding a non-singular behaviour. At infinity, the corresponding velocity field grows at a logarithmic rate. The associated pressure gradient may be found readily by substituting (2.26) in the Stokes equation enhanced with a singular forcing term that reflects the presence of the one-dimensional array.

3. Arrays of three-dimensional point forces: Beenakker's method

The slow convergence of the Fourier sums on the right-hand sides of (2.8) and (2.12) motivates the developments of alternative computational methods. Hasimoto [8] developed a method of computing the generating function S^{3D3P} and its Laplacian based on Ewald's original formulation. Ishii [9], and Sangani and Behl [7], discussed the analogous computation of S^{3D2P}. We find, however, that it is more straightforward to proceed in an alternative manner following Beenakker's [8] work for the Rotne–Prager tensor which, in turn, is motivated by an analogous formulation by Nijboer and De Wette [15] for the computation of generalized potentials in electrostatics.

3.1. Triply-periodic arrays

We begin by introducing the free-space Green's function or stokeslet representing the flow due to a three-dimensional point force located at the point $\mathbf{x}_0$ in infinite space, and express it in the form $\mathbf{S}(\mathbf{x}-\mathbf{x}_0) = (\mathbf{I}\nabla^2 - \nabla\nabla)r_0$ where $r_0 = |\mathbf{x}-\mathbf{x}_0|$ [1]. Next, we express the stokeslet in the form

$$\mathbf{S}(\mathbf{x}-\mathbf{x}_0) = \mathbf{\Theta}(\mathbf{x}-\mathbf{x}_0) + \mathbf{\Phi}(\mathbf{x}-\mathbf{x}_0), \tag{3.1}$$

where

$$\begin{bmatrix} \Theta \\ \Phi \end{bmatrix} (\mathbf{x}) = (\mathbf{I}\nabla^2 - \nabla\nabla) \begin{bmatrix} r \operatorname{erfc}(\xi r) \\ r \operatorname{erf}(\xi_r) \end{bmatrix}, \tag{3.2}$$

$r = |\mathbf{x}|$, and ξ is an arbitrary positive constant with dimensions of inverse length. The reason for this decomposition will be discussed shortly in this section. After some straightforward algebra we obtain

$$\Theta(\mathbf{x}) = \mathbf{I}\frac{C(\xi r)}{r} + \mathbf{xx}\frac{D(\xi r)}{r^3}, \tag{3.3}$$

where

$$\begin{aligned} C(x) &= \operatorname{erfc}(x) + \frac{2}{\sqrt{\pi}}(2x^2 - 3)x\exp(-x^2), \\ D(x) &= \operatorname{erfc}(x) + \frac{2}{\sqrt{\pi}}(1 - 2x^2)x\exp(-x^2). \end{aligned} \tag{3.4}$$

Now, we consider summing the right-hand side of (3.1) over all point forces. Noting that Θ decays in a Gaussian manner as the observation point $\mathbf{x}$ moves far from the location of the point force $\mathbf{x}_0$, suggests that it may be summed efficiently over a truncated range of point forces in the periodic lattice. The second component, Θ, does not decay fast enough to be summed over all point forces. To circumvent this difficulty, we introduce Poisson's summation formula, which is a special case of Parseval's identity stating that for any function F defined over the nodes of a three-dimensional lattice,

$$\sum_{n=0}^{\infty} F(\mathbf{X}_n) = \frac{1}{\tau}\sum_{\lambda=0}^{\infty} \hat{F}(\mathbf{k}_\lambda), \tag{3.5}$$

where $\hat{F}(\mathbf{k})$ is the three-dimensional Fourier transform of F with respect to $\mathbf{X}$, defined as

$$\hat{F}(\mathbf{k}) = \int_{R^3} \exp(i\mathbf{k}\cdot\mathbf{X})F(\mathbf{X})\,\mathrm{d}^3\mathbf{X}. \tag{3.6}$$

Note that (2.5) arises from (3.5) by identifying F with the three-dimensional delta function.

To apply (3.5) for $\mathbf{\Phi}$, we require its Fourier transform which, according to (3.6), is given by

$$\hat{\mathbf{\Phi}}(\mathbf{k}, \hat{\mathbf{x}}_0) = \int_{R^3} \exp(i\mathbf{k}\cdot\mathbf{X})\mathbf{\Phi}(\hat{\mathbf{x}}_0 - \mathbf{X})\,\mathrm{d}^3\mathbf{X}. \tag{3.7}$$

Introducing the explicit form of $\mathbf{\Phi}$ from (3.2) we obtain

$$\begin{aligned} \hat{\mathbf{\Phi}}(\mathbf{k}, \hat{\mathbf{x}}_0) &= \int_{R^3} \exp(i\mathbf{k}\cdot\mathbf{X})(\mathbf{I}\nabla^2 - \nabla\nabla)[r \operatorname{erf}(\xi r)]\,\mathrm{d}^3\mathbf{X} \\ &= (-\mathbf{I}|k|^2 + \mathbf{kk})\int_{R^3} \exp(i\mathbf{k}\cdot\mathbf{X})r \operatorname{erf}(\xi r)\,\mathrm{d}^3\mathbf{X}, \end{aligned} \tag{3.8}$$

where $r = |\mathbf{x} - \mathbf{x}_0 - \mathbf{X}|$. To evaluate the integral on the right-hand side of (3.8), call it Q, we work in spherical polar coordinates, writing

$$\begin{aligned} Q &= \exp(i\mathbf{k}\cdot\hat{\mathbf{x}}_0)\int_{R^3} \exp[i\mathbf{k}\cdot(\mathbf{X} - \hat{\mathbf{x}}_0)]r \operatorname{erf}(\xi r)\,\mathrm{d}^3\mathbf{X} \\ &= 2\pi \exp(i\mathbf{k}\cdot\hat{\mathbf{x}}_0)\int_0^\infty \int_0^\pi \exp(i|\mathbf{k}|r\cos\theta)\operatorname{erf}(\xi r)r^3 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r. \end{aligned} \tag{3.9}$$

Performing the integration with respect to the polar angle θ we obtain

$$\begin{aligned} Q &= \exp(i\mathbf{k}\cdot\hat{\mathbf{x}}_0)\frac{4\pi}{|\mathbf{k}|}\int_0^\infty \sin(|\mathbf{k}|r)\,\mathrm{erf}(\xi r)r^2\,\mathrm{d}r \\ &= -\exp(i\mathbf{k}\cdot\hat{\mathbf{x}}_0)\frac{4\pi}{|\mathbf{k}|}\frac{\partial^2}{\partial|\mathbf{k}|^2}\int_0^\infty \sin(|\mathbf{k}|r)\,\mathrm{erf}(\xi r)\,\mathrm{d}r. \end{aligned} \tag{3.10}$$

To compute the last integral in (3.10) we write

$$\begin{aligned} \int_0^\infty \sin(|\mathbf{k}|r)\,\mathrm{erf}(\xi r)\,\mathrm{d}r &= \int_0^\infty \sin(|\mathbf{k}|r)\,\mathrm{d}r - \int_0^\infty \sin(|\mathbf{k}|r)\,\mathrm{erfc}(\xi r)\,\mathrm{d}r \\ &= \frac{1}{|\mathbf{k}|}\exp\left(-\tfrac{1}{4}\omega^2\right), \end{aligned} \tag{3.11}$$

where we have defined $\omega = |\mathbf{k}|/\xi$. The third integral in (3.11) is computed using standard tables [13, p. 480]. The second integral is regarded as the Fourier transform of the step function, and is thus computed in terms of the Fourier transform of the one-dimensional delta function using the fact that the delta function is the derivative of the step function; the result is $1/|\mathbf{k}|$. Finally, we substitute (3.11) in (3.10) and then into (3.8), obtaining

$$\hat{\Phi}_{jm}(\hat{\mathbf{x}}_0,\mathbf{k}) = \frac{8\pi}{|\mathbf{k}|^2}\left(\delta_{jm} - \frac{k_j k_m}{|\mathbf{k}|^2}\right)\left(1 + \tfrac{1}{4}\omega^2 + \tfrac{1}{8}\omega^4\right)\exp\left(-\tfrac{1}{4}\omega^2\right)\exp(i\mathbf{k}\cdot\hat{\mathbf{x}}_0). \tag{3.12}$$

Combining (3.1), (3.5), and (3.12) we obtain the Green's function in the form of two exponentially decaying (and thus rapidly convergent) sums, as

$$G_{jm}^{3D3P}(\hat{\mathbf{x}}_0) = \sum_n \Theta_{jm}(\hat{\mathbf{x}}_n) + \frac{8\pi}{\tau}\sum_{\substack{\lambda \\ |\mathbf{k}_\lambda|\neq 0}}\left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m}\right) S^{3D3P-1}(\mathbf{k}_\lambda,\hat{\mathbf{x}}_0) \tag{3.13}$$

where

$$S^{3D3P-1}(\mathbf{k},\mathbf{x}) = \frac{1}{\xi^4}\left(\frac{1}{\omega^4} + \frac{1}{4}\frac{1}{\omega^2} + \frac{1}{8}\right)\exp\left(-\tfrac{1}{4}\omega^2\right)\exp(i\mathbf{k}\cdot\mathbf{x}) \tag{3.14}$$

[1]. The omission of the zero wave-number in the second sum is justified by the introduction of a mean pressure gradient associated with a three-dimensional lattice of point forces.

Decreasing the value of ξ renders the contribution from the sum in reciprocal space increasingly smaller compared to that of the first sum in physical space. As ξ tends to infinity, ω tends to zero, the first sum makes a vanishing contribution, the generating function tends to that shown in (2.9), and (3.13) reduces to Hasimoto's Fourier series shown in (2.8). There is an optimal value of ξ that minimizes the computational effort; for the simple cubic lattice Beenakker [10] recommends $\xi = \pi^{1/2}/\tau^{1/3}$; more generally, the optimal value depends upon the expense of computing or approximating the error function.

It is instructive to contrast the Gaussian decay of (3.14) with the algebraic quadratic decay of (2.9). By introducing the decomposition (3.1), we shift the singular behaviour of the Green's function near the point forces to the first component Θ. As a result, the Fourier series of the second component Φ decays an exponential rate, thus expediting the numerical computation.

3.2. DOUBLY-PERIODIC ARRAYS

To obtain a planar doubly-periodic array of three-dimensional point forces deployed in the plane of the base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, we work as discussed in the text surrounding Eq. (2.11), and find that (3.13) reduces to

$$G_{jm}^{3D2P}(\hat{\mathbf{x}}_0) = \sum_n \Theta_{jm}(\hat{\mathbf{x}}_n) + \frac{4\pi}{A} \sum_{\substack{\lambda \\ |\mathbf{l}_\lambda| \neq 0}} \left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m} \right) S^{3D2P-1}(\mathbf{l}_\lambda, \hat{\mathbf{x}}_0)$$
$$+ \frac{4\pi}{A} \left[\left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m} \right) (S^{3D2P-1} - S^{3D2P}) \right]_{|\mathbf{l}_\lambda|=0}, \tag{3.15}$$

where the first summation is over the planar array, $\mathbf{l}_\lambda = j_1\mathbf{b}_1 + j_2\mathbf{b}_2$ describes the two-dimensional reciprocal lattice which lies in the plane of the physical lattice, and the generating function is given by

$$S^{3D2P-1}(\mathbf{l}, \mathbf{x}) = \frac{1}{\pi\xi^4} \int_{-\infty}^{\infty} \left(\frac{1}{\omega^4} + \frac{1}{4}\frac{1}{\omega^2} + \frac{1}{8} \right) \exp\left[-\tfrac{1}{4}\omega^2 - i(\mathbf{l} + \eta\mathbf{e}_3) \cdot \mathbf{x} \right] \mathrm{d}\eta, \tag{3.16}$$

where $\omega = |\mathbf{l} + \eta\mathbf{e}_3|/\xi$. It is important to note that excluding the generating function S^{3D3P} given in (2.13) from the summation over the null wave number $\mathbf{l}_\lambda$ on the right-hand side of (3.15) is consistent with our earlier action in Section 2. Straightforward rearrangement of (3.16) yields

$$S^{3D2P-1}(\mathbf{l}, \mathbf{x}) = \frac{2}{\pi\xi^3} \left[I_0(\delta, \zeta) + \tfrac{1}{4} I_1(\delta, \zeta) + I_2(\delta, \zeta) \right] \exp\left(-\tfrac{1}{4}\zeta^2 - i\mathbf{l} \cdot \mathbf{x} \right), \tag{3.17}$$

where $\delta = \xi\mathbf{e}_3 \cdot \mathbf{x}$, $\zeta = |\mathbf{l}|/\xi$, and we have defined the integrals

$$I_n(\delta, \zeta) = \int_0^{\infty} \frac{\cos(\delta t)}{(\zeta^2 + t^2)^{2n}} \exp\left(-\tfrac{1}{4}t^2 \right) \mathrm{d}t. \tag{3.18}$$

Reference to tables of definite integrals yields

$$I_0 = \pi^{1/2} \exp(-\delta^2), \tag{3.19a}$$

$$I_1(\delta, \zeta) = -\frac{\pi}{4}\frac{1}{\zeta} \exp\left(\tfrac{1}{4}\zeta^2 \right) \left[\mathrm{e}^{-\rho} \operatorname{erf}\left(\tfrac{1}{2}\zeta - |\delta| \right) + \mathrm{e}^{\rho} \operatorname{erf}\left(\tfrac{1}{2}\zeta + |\delta| \right) - 2\cosh(\rho) \right], \tag{3.19b}$$

$$I_2(\delta, \zeta) = -\frac{1}{2\zeta} \frac{\partial I_1(\delta, \zeta)}{\partial \zeta}, \tag{3.19c}$$

where $\rho \equiv |\delta|\ \zeta = |\mathbf{l}|\ |\mathbf{e}_3 \cdot \mathbf{x}|$, and the error function satisfies the symmetry condition $\operatorname{erf}(x) = -\operatorname{erf}(-x)$ [13, p. 480; 14, p. 453].

In the limit as ξ tends to infinity while $|\mathbf{l}|$ and $\mathbf{e}_3 \cdot \mathbf{x}$ are held constant, the integrals I_0, I_1 and I_2 exhibit the asymptotic behaviour

$$I_0(\delta, \zeta) \approx 0, \quad I_1(\delta, \zeta) \approx \frac{\pi}{2}\frac{\mathrm{e}^{-\rho}}{\zeta}, \quad I_2(\delta, \zeta) \approx \frac{\pi}{4}\frac{\mathrm{e}^{-\rho}}{\zeta^3}. \tag{3.20}$$

In this case (3.17) reduces to (2.13), yielding the Green's function in terms of a Fourier series.

Both generating functions (3.17) and (2.13) decay at an exponential rate with respect to ρ. As $|\mathbf{l}|$ tends to infinity while ξ and $\mathbf{e}_3 \cdot \mathbf{x}$ are constant, however, the factor that multiplies the exponential term in (2.13) decays at a power-law rate with respect to $|\mathbf{l}|$, whereas that in (3.17) decays at a Gaussian rate. This feature reduces the computational cost by a substantial amount, especially for small values of ρ.

To obtain the Green's function we require the first and second partial derivatives of the generating function with respect to ζ. These are given by long expressions which are available by the author at the reader's request. In practice, it is more expedient to compute these derivatives by numerical differentiation.

4. Arrays of three-dimensional point forces: second fast-summation method

The presence of the error function and the complexity of the generating function for flow due to a two-dimensional array of three-dimensional point forces given in (3.17) motivates the search of alternative methods in hopes of efficient numerical computation. With this goal in mind, we begin afresh with a different kind of splitting between real and reciprocal space.

4.1. Triply-periodic arrays

Replacing the error function in (3.2) with an exponentially decaying function we obtain the alternative form

$$\begin{bmatrix} \Theta \\ \Phi \end{bmatrix} (\mathbf{x}) = (\mathbf{I}\nabla^2 - \nabla\nabla) \begin{bmatrix} r\exp(-\xi r) \\ r[1-\exp(-\xi r)] \end{bmatrix}, \tag{4.1}$$

where $r = |\mathbf{x}|$ and, as before, ξ is a positive constant with dimensions of inverse length. After some straightforward algebra we find that Θ is given by (3.3) with

$$C(x) = (1 - 3x + x^2)\mathrm{e}^{-x}, \qquad D(x) = (1 + x - x^2)\mathrm{e}^{-x}. \tag{4.2}$$

Furthermore, working as in Section 3, we find that the Green's function is given by (3.13), where the generating function assumes the new form

$$S^{3D3P-2} = \frac{1}{\xi^4}\left(\frac{1}{\omega^4} + \frac{3}{\omega^2} + 6\right)\frac{1}{\omega^2+1}\exp(i\mathbf{k}\cdot\mathbf{x}). \tag{4.3}$$

The exponential decay of the modulating functions C and D in (4.2) may be contrasted with the Gaussian decay of the corresponding functions of (3.4). Furthermore, the sixth-order algebraic decay of the generating function in (4.3) may be contrasted with the Gaussian decay of (3.14) but also, with the fourth-order algebraic decay of (2.9). Clearly, the present method is an improvement over the Fourier-series method. Computational cost for evaluating the error function aside, the method described in Section 3 leads to faster summation. The advantages of the present method will become evident when we consider the Green's functions for two-dimensional planar arrays.

4.2. Doubly-periodic arrays

Working as in Section 3, we find that the Green's function for a doubly-periodic array is given by (3.15), where the generating function assumes the new form

$$S^{3D2P-2}(\mathbf{l},\mathbf{x}) = \frac{1}{\pi\xi^4}\int_{-\infty}^{\infty} \frac{6\omega^4 + 3\omega^2 + 1}{\omega^4(\omega^2+1)^3}\exp[-i(\mathbf{l} + \eta\mathbf{e}_3)\cdot\mathbf{x}]\,\mathrm{d}\eta \tag{4.4}$$

with $\omega = |\mathbf{l}+\eta \mathbf{e}_3|/\xi$. The integral in (4.4) may be computed in closed form by means of contour integration in the complex plane. For example, when $\mathbf{e}_3 \cdot \mathbf{x} < 0$, we introduce a semi-circular contour in the upper half-plane that joins the real axis to form a closed contour, and note that the integrand in (4.4) has a double pole at $\eta = i|\mathbf{l}|$, and a triple pole at $\eta = i(\xi^2 + |\mathbf{l}|^2)^{1/2}$. As the radius of the semi-circular contour tends to infinity, the corresponding integral vanishes, and the integral in (4.4) becomes equal to the sum of the residues multiplied by $2\pi i$. The residues were computed using the program Mathematica©, programmed by Chad Coulliette; a copy of the script is available on request from the author. The result may be placed in the form

$$S^{3D2P-2}(\mathbf{l},\mathbf{x}) = S^{3D2P}(\mathbf{l},\mathbf{x}) + \frac{Q(\delta,\zeta)}{2\xi^3}\exp(-i\mathbf{l}\cdot\mathbf{x}), \tag{4.5}$$

where

$$Q(\delta,\zeta) = \frac{2-\zeta^2+\delta(2-\zeta^2)\sqrt{1+\zeta^2}+\delta^2(1+\zeta^2)}{(1+\zeta^2)^{5/2}}\exp\left(-\delta\sqrt{1+\zeta^2}\right), \tag{4.6}$$

$\zeta = |\mathbf{l}|/\xi$, $\rho = |\mathbf{l}|\ |\mathbf{e}_3 \cdot \mathbf{x}|$, and $\delta = \xi|\mathbf{e}_3 \cdot \mathbf{x}|$. An identical expression is obtained when $\mathbf{e}_3 \cdot \mathbf{x} > 0$.

The first term on the right-hand side of (4.5) is the generating function corresponding to the Fourier-series solution given in (2.13). In the limit as ξ tends to infinity while $|\mathbf{l}|$ and $\mathbf{e}_3 \cdot \mathbf{x}$ are held constant, the second term on the right-hand side of (4.5) vanishes yielding the Green's function in the form of a pure Fourier series. On the other hand, as $|\mathbf{l}|$ tends to infinity while ξ and $\mathbf{e}_3 \cdot \mathbf{x}$ are held constant, (4.5) assumes the asymptotic form

$$S^{3D2P-2}(\mathbf{l},\mathbf{x}) \approx \tfrac{1}{2}\xi^2 \frac{\rho^2}{|\mathbf{l}|^5}\exp(-i\mathbf{l}\cdot\mathbf{x} - \rho). \tag{4.7}$$

Both (4.7) and (2.13) decay at an exponential rate with respect to ρ. The power of $|\mathbf{l}|$ that multiplies the exponential in (4.7), however, is higher by two units than that of (2.13), and this expedites the computations, especially for small values of ρ.

The first term on the right-hand side of (4.5) is summed over all lattice points $\mathbf{l}_\lambda$ with the exception of the null point $|\mathbf{l}_\lambda| = 0$, in agreement with our earlier discussion in Section 2. The second term is summed over all lattice points, *including* the null point $|\mathbf{l}_\lambda| = 0$; the value of the second term at the null point is readily found to be $[1/(2\xi^3)](2+2\delta+\delta^2)\exp(-\delta)$. In the limit as ξ tends to infinity, the contribution of this term tends to vanish.

To obtain the Green's function we require the first and second partial derivatives of the generating function with respect to δ. These are given by relatively simple but rather long expressions which can be obtained from the author on request. In practice, it is more expedient to compute these derivatives by numerical differentiation.

5. Arrays of two-dimensional point forces: Fourier-series solution

In the final two sections we consider doubly- and singly-periodic arrays of two-dimensional point forces following the general discussion of the preceding sections for three-dimensional flow.

5.1. Doubly-periodic arrays

We begin by considering the flow due to a two-dimensional lattice of two-dimensional point forces in the xy plane. One of the point forces is located at the point $\mathbf{x}_0$ and the rest of them are located at points that are separated by the lattice vectors $\mathbf{X}_n = i_1\mathbf{a}_1 + i_2\mathbf{a}_2$ where $\mathbf{a}_1, \mathbf{a}_2$ are two arbitrary base vectors lying in the xy plane, and i_1, i_2 are two integers. The associated velocity field and pressure fields satisfy the continuity equation $\nabla \cdot \mathbf{u} = 0$ and the periodically forced two-dimensional version of the Stokes equation (2.1), where δ_3 is replaced by the two-dimensional delta function δ_2. Following standard convention, we express the solution in terms of the periodic Green's function $\mathbf{G}^{2D2P}$ and corresponding pressure vector $\mathbf{p}^{2D2P}$ defined by $u_j = 1/(4\pi\mu)G_{jm}^{2D2P}g_m$ and $p = 1/(4\pi)p_m^{2D2P}g_m$, and obtain

$$-\frac{\partial p_m^{2D2P}}{\partial x_j} + \nabla^2 G_{jm}^{2D2P} + 4\pi\delta_{jm}\sum_n \delta_2(\hat{\mathbf{x}}_n) = 0, \qquad \frac{\partial G_{jm}^{2D2P}}{\partial x_j} = 0. \tag{5.1}$$

Exploiting the periodicity of the flow, we expand $\mathbf{G}^{2D2P}$ and $\nabla\mathbf{p}^{2D2P}$ in complex Fourier series as

$$G_{jm}^{2D2P} = \sum_\lambda^\infty \hat{G}_{jm\lambda}^{2D2P} \exp(-i\mathbf{l}_\lambda \cdot \hat{\mathbf{x}}_0), \qquad \frac{\partial p_m^{2D2P}}{\partial x_j} = \sum_\lambda^\infty \hat{p}_{jm\lambda}^{2D2P} \exp(-i\mathbf{l}_\lambda \cdot \hat{\mathbf{x}}_0), \tag{5.2}$$

where $\mathbf{l}_\lambda = j_1\mathbf{b}_1 + j_2\mathbf{b}_2$ are the reciprocal lattice points, j_1, j_2 are two integers, $\mathbf{b}_1, \mathbf{b}_2$ are the reciprocal base vectors defined as

$$\mathbf{b}_1 = \frac{2\pi}{A}\mathbf{a}_2 \times \mathbf{e}_z, \qquad \mathbf{b}_2 = \frac{2\pi}{A}\mathbf{e}_z \times \mathbf{a}_1. \tag{5.3}$$

$A = |\mathbf{a}_1 \times \mathbf{a}_2|$ is the area of one periodic cell, and $\mathbf{e}_z$ is the unit vector in the direction of the z axis. The physical and reciprocal lattice points satisfy the equation $\mathbf{X}_n \cdot \mathbf{l}_\lambda = 2\pi m$ where m is an integer. Furthermore, we introduce identity (2.15), and substitute (5.2) in (5.1) to derive the algebraic system of equations

$$-\hat{p}_{jm\lambda}^{2D2P} - |\mathbf{l}_\lambda|^2 \hat{G}_{jm\lambda}^{2D2P} + \frac{4\pi}{A}\delta_{jm} = 0, \qquad l_{j\lambda}\hat{G}_{jm\lambda}^{2D2P} = 0. \tag{5.4}$$

When $|\mathbf{l}_0| = 0$ the solution is

$$\hat{G}_{jm0}^{2D2P} = 0, \qquad \hat{p}_{jm0}^{2D2P} = \frac{4\pi}{A}\delta_{jm}, \tag{5.5a}$$

whereas, when $|\mathbf{l}_\lambda| \neq 0$, the solution is

$$\hat{G}_{jm\lambda}^{2D2P} = \frac{4\pi}{A}\frac{1}{|\mathbf{l}_\lambda|^2}\left(\delta_{jm} - \frac{l_{j\lambda}l_{m\lambda}}{|\mathbf{l}_\lambda|^2}\right), \qquad \hat{p}_{jm\lambda}^{2D2P} = \frac{4\pi}{A}\frac{l_{j\lambda}l_{m\lambda}}{|\mathbf{l}_\lambda|^2}. \tag{5.5b}$$

Substituting (5.5a,b) in (5.2) we obtain the Green's function in the form

$$\begin{aligned} G_{jm}^{2D2P} &= \frac{4\pi}{A}\sum_\lambda \frac{1}{|\mathbf{l}_\lambda|^2}\left(\delta_{jm} - \frac{l_{j\lambda}l_{m\lambda}}{|\mathbf{l}_\lambda|^2}\right)\exp(-i\mathbf{l}_\lambda \cdot \hat{\mathbf{x}}_0) \\ &= \frac{4\pi}{A}\sum_\lambda \left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m}\right) S^{2D2P}(\mathbf{l}_\lambda, \hat{\mathbf{x}}_0) \end{aligned} \tag{5.6}$$

where the generating function is given by

$$S^{2D2P}(\mathbf{l}, \mathbf{x}) = \frac{\exp(-i\mathbf{l} \cdot \mathbf{x})}{|\mathbf{l}|^4}. \tag{5.7}$$

The associated pressure gradient is given by

$$\frac{\partial p_m^{2D2P}}{\partial x_j} = \frac{4\pi}{A}\delta_{jm} + \frac{4\pi}{A}\sum_{\substack{\lambda \\ |\mathbf{l}_\lambda|\neq 0}} \frac{l_{j\lambda} l_{m\lambda}}{|\mathbf{l}_\lambda|^2}\exp(-i\mathbf{l}_\lambda\cdot\hat{\mathbf{x}}_0)$$
$$= \frac{4\pi}{A}\delta_{jm} - \frac{4\pi}{A}\sum_{\substack{\lambda \\ |\mathbf{l}_\lambda|\neq 0}} \frac{\partial}{\partial x_j \partial x_m}\nabla^2 S^{2D2P}(\mathbf{l}_\lambda, \hat{\mathbf{x}}_0). \tag{5.8}$$

5.2. SINGLY-PERIODIC ARRAY

To obtain a singly-periodic array of three-dimensional point forces arranged in the direction of the base vector $\mathbf{a}_1$, we set $\mathbf{a}_2$ perpendicular to $\mathbf{a}_1$, and write $A = aL$ where $a = |\mathbf{a}_1|$ is the separation of the point forces and $L = |\mathbf{a}_2|$. The reciprocal base vectors are given in the first two equations of (2.11). In the limit, as L tends to infinity, Eq. (5.7) yields

$$G_{jm}^{2D1P} = \frac{2\pi}{a}\sum_{\substack{\lambda \\ |\mathbf{m}_\lambda|\neq 0}} \left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m}\right) S^{2D1P}(\mathbf{m}_\lambda, \hat{\mathbf{x}}_0), \tag{5.9}$$

where $\mathbf{m}_\lambda = \lambda \mathbf{b}_1$ are the reciprocal lattice points, λ is an integer, and the generating function is given by

$$S^{2D1P}(\mathbf{m},\mathbf{x}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\exp[-i(\mathbf{m}+\eta\mathbf{e}_2)\cdot\mathbf{x}]}{|\mathbf{m}+\eta\mathbf{e}_2|^4}\, d\eta = \tfrac{1}{2}\frac{1+\rho}{|\mathbf{m}|^3}\exp(-i\mathbf{m}\cdot\mathbf{x} - \rho), \tag{5.10}$$

where $\rho = |\mathbf{m}|\, \mathbf{e}_2\cdot\mathbf{x}$. It is instructive to note that (5.9) and (5.10) may also be obtained by integrating the generating function for the double-periodic array of three-dimensional point forces (2.13) with respect to the z coordinate of the point sources over one period.

The corresponding pressure gradient is found by enhancing the first term on the right-hand side of (5.8) with a one-dimensional delta function δ_1 acting in the plane of the one-dimensional array, and integrating the last term as in (5.11). The result is

$$\frac{\partial p_m^{2D1P}}{\partial x_j} = \frac{2\pi}{a}\delta_{jm}\delta_1(\hat{\mathbf{x}}_{02}) - \frac{2\pi}{a}\sum_{\substack{\lambda \\ |\mathbf{m}_\lambda|\neq 0}} \frac{\partial}{\partial x_j \partial x_m}\nabla^2 S^{3D2P}(\mathbf{m}_\lambda, \hat{\mathbf{x}}_0). \tag{5.11}$$

In deriving the singly-periodic from the double-periodic array, we abandon the summation over the array corresponding to the null wave number $|\mathbf{m}_\lambda| = 0$ as shown in (5.9) and (5.11). As a result, the flow vanishes at an exponential rate far from the array, and the pressure assumes a uniform value.

One may readily verify that the Laplacian of the generating function given in (5.10) represents the steady temperature field due to the combined action of a singly-periodic array of two-dimensional point sources of heat situated at the point forces, and a continuous distribution of corresponding point sinks situated in the line of the array, so that the net rate of transport of heat across a period vanishes. The distributed singularities cause the stress field associated with the Green's function to exhibit a singular behaviour at the line of the point forces, which is analogous to that of the doubly-periodic array of three-dimensional point force discussed in Section 2.

When the point forces are oriented normal to the array, in order to obtain a Green's function with a continuous stress field we simply enhance the pressure with an appropriate step function. When the point forces are tangential to the array, we introduce a symmetric shear flow that cancels the discontinuity in shear stress, thus obtaining the *regularized* form

$$G_{11}^{2D1P-R} = G_{11}^{2D1P} - \frac{2\pi}{a}|\hat{x}_{0_2}|. \tag{5.12}$$

Fortunately, the regularized Green's function may be computed closed-form solution rendering the above derivation a mere academic alternative [1,17].

6. Arrays of two-dimensional point forces: fast summation methods

The slow converge of the partial sums on the right-hand sides of (5.6) and (5.8) renders the Fourier expansion uneconomical for numerical computation. Summation in closed form is discouraged by the occurrence of Jacobi's theta functions in the double summation of the logarithmic kernel [18]. To circumvent these difficulties, we explore alternative formulations based on Ewald's and related summation methods.

6.1. Hasimoto's method

Hasimoto [8] developed a method for computing the generating function (5.7) and its Laplacian based on Ewald's original formulation. The final results, given explicitly by van de Vorst [11], may be placed in the form

$$G_{jm}^{2D2P}(\mathbf{x}, \mathbf{x}_0) = \sum_n \Psi_{jm}(\hat{\mathbf{x}}_n) + \frac{4\pi}{A} \sum_{\substack{\lambda \\ |\mathbf{l}_\lambda| \neq 0}} \left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m} \right) S^{2D2P-1}(\mathbf{l}_\lambda, \hat{\mathbf{x}}_0), \tag{6.1}$$

where

$$\Psi_{jm}(\mathbf{x}) = -\delta_{jm} P(\xi r) + \frac{x_j x_m}{r^2} Q(\xi r). \tag{6.2}$$

$r = |\mathbf{x}|$, ξ is a positive parameter, $P(x) = -(1/2)E_1(x^2) + \exp(-x^2)$, and $Q(x) = \exp(-x^2)$. The exponential E_1 may be computed efficiently using polynomial or rational approximations [14, p. 231]. As ξ tends to infinity, the first sum in (6.1) makes a vanishing contribution and we recover the Green's function in terms of the two-dimensional Fourier series given in (5.7). As x tends to vanish, $E_1(x^2)$ behaves like $-2\ln x$, $C(x)$ behaves like $\ln x$, $D(x)$ tends to unity, and the corresponding term in the first sum of (6.1) reduces to the two-dimensional Stokeslet.

The generating function S^{2D2P-1} within the second sum in (6.1) is given by

$$S^{2D2P-1}(\mathbf{l}, \mathbf{x}) = \frac{1}{\xi^4}\left(\frac{\xi^4}{|\mathbf{l}|^4} + \frac{1}{4}\frac{\xi^2}{|\mathbf{l}|^2} \right) \exp\left(-\frac{1}{4}\frac{\xi^2}{|\mathbf{l}|^2} \right) \exp(i\mathbf{l} \cdot \mathbf{x}). \tag{6.3}$$

Following the discussion of Section 2 we find that the flow rate across an infinite line that is perpendicular to one of the base lattice vectors is equal to zero.

6.2. Second fast-summation method

Another way of deriving the doubly-periodic periodic Green's function is to begin from the three-dimensional triply-periodic array discussed in Section 2, require that $\mathbf{a}_1$ and $\mathbf{a}_2$ lie in the

xy plane, and set $\mathbf{a}_3 = L\mathbf{e}_z$ where $\mathbf{e}_z$ is the unit vector along the z axis and L is an arbitrary period. The reciprocal base vectors are given in (2.11) with $\mathbf{e}_3 = \mathbf{e}_z$. Stipulating that both the observation point $\mathbf{x}$ and the location of the point force $\mathbf{x}_0$ lie in the xy plane, we integrate (2.26) with respect to z_0 over one period of the three-dimensional lattice from $z_0 = 0$ to $z_0 = L$, finding

$$G_{jm}^{2D2P}(\mathbf{x},\mathbf{x}_0) = \tfrac{1}{2}\sum_n \int_0^L \Theta_{jm}(\hat{\mathbf{x}}_n)\,\mathrm{d}z_0 + \frac{4\pi}{A}\sum_{\substack{\lambda \\ |\mathbf{l}_\lambda|\neq 0}} \left(-\delta_{jm}\nabla^2 + \frac{\partial}{\partial x_j \partial x_m}\right) S^{2D2P-2}(\mathbf{l}_\lambda, \hat{\mathbf{x}}_0), \tag{6.4}$$

where the generating function is given by

$$S^{2D2P-2}(\mathbf{l},\mathbf{x}-\mathbf{x}_0) = \frac{1}{L}\int_0^L S^{3D3P-1}(\mathbf{l},\mathbf{x}-\mathbf{x}_0)\,\mathrm{d}z_0. \tag{6.5}$$

Note that the terms on the right-hand side of (6.4) have been multiplied by a factor of one-half due to the standard convention in the definition of the two-dimensional Green's function with respect to the strength of a two-dimensional point force.

Substituting the definition of Θ from (3.3) in the first integral on the right-hand side of (6.5) we obtain

$$\tfrac{1}{2}\int_0^L \Theta(\mathbf{x}-\mathbf{x}_0)\,\mathrm{d}z_0 = [E(\xi, r_0)\mathbf{I} + (\mathbf{x}-\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)H(\xi, r_0)], \tag{6.6}$$

where $r_0 = |\mathbf{x}-\mathbf{x}_0|$, the functions E and H are given by

$$E(\xi, r) = \int_0^\infty \frac{C\left(\xi\sqrt{r^2+z^2}\right)}{(r^2+z^2)^{1/2}}\,\mathrm{d}z, \qquad H(\xi, r) = \int_0^\infty \frac{D\left(\xi\sqrt{r^2+z^2}\right)}{(r^2+z^2)^{3/2}}\,\mathrm{d}z \tag{6.7}$$

and the functions C and D are given in (3.4). A formal asymptotic expansion shows that, in the limit as r tends to zero, E behaves like $-\ln r$ whereas H behaves like r^{-2}, in which case the right-hand side of (6.6) reduces to the free-space two-dimensional stokeslet [6]. On the other hand, as r tends to infinity, the functions C and D, and thus E and H, decay in a Gaussian manner.

Unfortunately, it appears that neither of the integrals in (6.7) may be evaluated in closed form. Fortunately, both integrals may be evaluated by a relatively simple and efficient numerical method. To compute the function E, we note that the integrand is singular when $r = 0$, and $C(0) = 1$, and express it in the form

$$E(\xi, r) = \int_0^{N(\xi)} \frac{C\left(\xi\sqrt{r^2+z^2}\right) - 1}{(r^2+z^2)^{1/2}}\,\mathrm{d}z + \ln\frac{N+\sqrt{r^2+N^2}}{r}, \tag{6.8}$$

where N is a specified truncated limit of integration whose optimal value is adjusted by numerical experimentation. The integral in (6.8) is non-singular and may be computed using a standard numerical method, such as a Gauss-Legendre quadrature [4]. A similar method may be devised for the computation of the function H. Constructing tables of E and H and then computing their values by interpolation, in particular, results in a most economical strategy of computation [5].

To obtain the generating function S^{2D2P-2} we integrate (3.14) over z_0 as indicated in (6.5), and find that the integral vanishes when $k_3 \neq 0$. Carrying out the integration for $k_3 = 0$ yields

$$S^{2D2P-2}(\mathbf{l}, \mathbf{x}) = \frac{1}{\xi^4}\left(\frac{\xi^4}{|\mathbf{l}|^4} + \frac{1}{4}\frac{\xi^2}{|\mathbf{l}|^2} + \frac{1}{8}\right)\exp\left(-\frac{1}{4}\frac{\xi^2}{|\mathbf{l}|^2}\right)\exp(i\mathbf{l}\cdot\mathbf{x}), \tag{6.9}$$

which differs from the function S^{2D2P-1} shown in (6.3) only by the presence of the constant 1/8 within the first set of parentheses on the right-hand side. Substituting (6.9) along with (6.6) in (6.4) gives the final form

$$\begin{aligned}\mathbf{G}^{2D2P}(\mathbf{x}, \mathbf{x}_0) = & \sum_k [\mathbf{I}E(\xi, |\hat{\mathbf{x}}_k|) + \hat{\mathbf{x}}_k\hat{\mathbf{x}}_k H(\xi, |\hat{\mathbf{x}}_k|)] \\ & + \frac{4\pi}{A}\sum_{\substack{\lambda \\ |\mathbf{l}_\lambda| \neq 0}} \cos[\mathbf{l}_\lambda \cdot (\mathbf{x} - \mathbf{x}_0)]\frac{1}{|\mathbf{l}_\lambda|^2}\left(\mathbf{I} - \frac{\mathbf{l}_\lambda\mathbf{l}_\lambda}{|\mathbf{l}_\lambda|^2}\right) \\ & \times \left[1 + \frac{1}{4}\left(\frac{|\mathbf{l}_\lambda|}{\xi}\right)^2 + \frac{1}{8}\left(\frac{|\mathbf{l}_\lambda|}{\xi}\right)^4\right]\exp\left[-\left(\frac{|\mathbf{l}_\lambda|}{2\xi}\right)^2\right].\end{aligned} \tag{6.10}$$

As expected, the right-hand side of (6.10) is independent of L. As ξ tends to infinity, the first sum makes a vanishing contribution and we recover the Green's function in terms of a two-dimensional Fourier series as in (5.6). Expression (6.1) and (6.10) are equivalent, but the former is more expedient for numerical computation.

7. Summary

Efficient numerical methods for computing the Green's functions representing the flow due to triply- and doubly-periodic arrays of three-dimensional point forces are available. The flow due to a singly-periodic array of three-dimensional point forces resits efficient numerical computation. The flow due to a doubly-periodic array of two-dimensional point forces can be computed efficiently using two alternative fast-summation methods, and that due to a singly-periodic array is available in closed form.

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